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# ROTATORY AND HORIZONTAL VIBRATIONS OF A CIRCULAR SURFACE FOOTING ON A SATURATED ELASTIC HALF-SPACE

M. K. KASSIR
The City College of City University of New York, Department of Civil Engineering, Convent Ave, New York, NY 10031, U.S.A.

and

J. XU and K. K. BANDYOPADYAY
Brookhaven National Laboratory, Department of Advanced Technologies, Upton, NY 11973, U.S.A.

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**Abstract**—The forced vibration of a liquid-filled, porous, elastic half-space produced by rotatory and horizontal oscillations of a circular surface footing is reduced to the solutions of Fredholm integral equations of the second kind. The circular footing is modeled by a pervious, weightless, rigid disk of negligible thickness. For a medium consisting of dense sand saturated by ground water, numerical solutions of the integral equations are obtained to reveal the variations of the impedance functions with the exciting frequency and the material parameters (permeability and Poisson's ratio). Comparison with the response of the dry soil is also discussed. In the rocking oscillation case, the presence of the ground water is found to influence the magnitude and shape of the impedance functions and may need to be considered in applicable soil–structure interaction problems. In the horizontal vibration case, however, marginal influence of the pore water is found to affect the response of the medium.

## 1. INTRODUCTION

The dynamic force–displacement relationship (ratio of applied force or couple/linear or angular displacement) plays a central role in determining the response of surface structures to dynamic loadings, in particular seismic excitation and machine vibration. For an elastic half-space indented by a rigid circular disk, an ever-increasing amount of research work has been devoted to the topic and comprehensive reviews are available in Gladwell (1968) and Luco and Westmann (1971). The fruitful method of solution which has evolved from this work is integral transform formulation of the problem in terms of auxiliary functions governed by Fredholm integral equations of the second kind. The integral equations are solved either numerically or by perturbation to determine the desired force–displacement relationship and other quantities of interest. When the half-space consists of a porous elastic solid filled with liquid, analogous solutions can be developed to determine the response to forced vibrations. Due to dissipation of the pore fluid, there is a coupling between the states of stress in the solid and fluid portions of the medium which could impact the response from the practical viewpoint. The response to normal excitation of the saturated medium by a circular disk has been determined by Kassir *et al.* (1989). The aim of this investigation is to determine the corresponding responses to rocking and horizontal excitations, and to find out to what degree the responses are influenced by the presence of the fluid. The steady-state responses of a square plate and a strip bearing on saturated elastic medium have been discussed by Halpren and Christiano (1986) and Kassir and Xu (1988).

Section 2 contains a brief summary of the equations governing the general propagation of waves in the two-phase medium. These equations were first formulated by Biot (1962). The set of equations governing the coupled dilatational and shear waves is given in Section 3. Section 4 contains a solution of the rocking excitation case (rotation about an axis parallel to the plane boundary) while the horizontal vibration case (translation parallel to an axis in the plane boundary) is discussed in Section 5. For both cases numerical results of the force–displacement relationships (stiffness and damping coefficients) are presented, and discussions on the impact of the fluid on the impedance functions over a wide range of values of the applied frequency are provided. For a medium consisting of dense sand saturated by ground water, it can be concluded that the influence of ground water in the frequency range of practical interest is important and, in the rocking oscillation case, needs to be considered in determining the response of surface footings to dynamic loading, especially earthquake loading.

## 2. BASIC EQUATIONS

Consider a pervious rigid circular disk of radius $a$ undergoing rocking and horizontal vibrations at the surface of a semi-infinite space consisting of a two-phase medium. The disk is assumed to be weightless. In terms of cylindrical coordinates $(r, \theta, z)$ located at the center of the disk with the $z$-axis pointing towards the medium, denote the displacement components of the solid material by $(u_r, u_\theta, u_z)$ and of the fluid part by $(U_r, U_\theta, U_z)$. The displacement components of the fluid relative to the solid, measured in terms of the volume per unit area of the bulk material, are $w_j = f\,(U_j - u_j)$, $j = r, \theta, z$, where $f$ stands for the porosity coefficient of the medium. The corresponding total (bulk) stresses are denoted by $\tau_{ij}$, $i, j = r, \theta, z$ and the pore pressure is denoted by $p$. With these notations the equations governing the propagation of waves are

$$\frac{\partial \tau_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_r - \tau_\theta}{r} = \rho \ddot{u}_r + \rho' \ddot{w}_r \tag{1a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = \rho \ddot{u}_\theta + \rho' \ddot{w}_\theta \tag{1b}$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_z}{\partial z} + \frac{\tau_{zr}}{r} = \rho \ddot{u}_z + \rho' \ddot{w}_z \tag{1c}$$

for the bulk stresses, and

$$-\frac{\partial p}{\partial r} = \rho \ddot{u}_r + \frac{\rho'}{f}\ddot{w}_r + \frac{\gamma'}{k}\dot{w}_r \tag{2a}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = \rho \ddot{u}_\theta + \frac{\rho'}{f}\ddot{w}_\theta + \frac{\gamma'}{k}\dot{w}_\theta \tag{2b}$$

$$-\frac{\partial p}{\partial z} = \rho \ddot{u}_z + \frac{\rho'}{f}\ddot{w}_z + \frac{\gamma'}{k}\dot{w}_z \tag{2c}$$

for the flow of fluid. In eqns (1) and (2), $\rho$, $\rho'$ stand for the mass densities of the composite solid–fluid parts, respectively. Also, $k$ and $\gamma'$ are Darcy's coefficient of permeability of the medium and the unit weight of the fluid. A dot over a letter indicates differentiation with respect to the time variable.

The pore pressure is given by the relation

$$p = \alpha'(-\alpha e + e'). \tag{3}$$

While the stress–strain relations assume the form

$$\tau_r = 2\mu\varepsilon_r + \lambda e - \alpha\alpha' e' \tag{4a}$$

$$\tau_\theta = 2\mu\varepsilon_\theta + \lambda e - \alpha\alpha' e' \tag{4b}$$

$$\tau_z = 2\mu\varepsilon_z + \lambda e - \alpha\alpha' e' \tag{4c}$$

$$\tau_{r\theta} = \mu\varepsilon_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) \tag{4d}$$

$$\tau_{\theta z} = \mu\varepsilon_{\theta z} = \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{4e}$$

$$\tau_{rz} = \mu\varepsilon_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{4f}$$

In eqns (3) and (4), $\lambda$ and $\mu$ are Lame's constants of the solid, and $\alpha$ and $\alpha'$ designate the compressibilities of the solid and fluid portions, respectively. The dilatations in the solid and fluid parts are denoted by $e$ and $e'$, i.e.

$$e = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial}{\partial z}(ru_z)\right] \tag{5a}$$

$$e' = -\frac{1}{r}\left[\frac{\partial}{\partial r}(rw_r) + \frac{\partial w_\theta}{\partial \theta} + \frac{\partial}{\partial z}(rw_z)\right]. \tag{5b}$$

The displacement equations of motion governing the propagation of waves in the medium can be obtained from relations (1)–(5). The results are

$$V_c^2\frac{\partial}{\partial r}(e-K\alpha e') + 2V_s^2\left(\frac{\partial \Omega_\theta}{\partial z} - \frac{1}{r}\frac{\partial \Omega_z}{\partial \theta}\right) = \ddot{u}_r + N\ddot{w}_r \tag{6a}$$

$$\frac{V_c^2}{r}\frac{\partial}{\partial \theta}(e-K\alpha e') + 2V_s^2\left(\frac{\partial \Omega_z}{\partial r} - \frac{\partial \Omega_r}{\partial z}\right) = \ddot{u}_\theta + N\ddot{w}_\theta \tag{6b}$$

$$V_c^2\frac{\partial}{\partial z}(e-K\alpha e') + 2V_s^2\left[\left(\frac{1}{r}\frac{\partial \Omega_r}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial r}(r\Omega_\theta)\right)\right] = \ddot{u}_z + N\ddot{w}_z \tag{6c}$$

$$KV_c^2\frac{\partial}{\partial r}(\alpha e - e') = N\ddot{u}_r + \frac{N}{f}\ddot{w}_r + \frac{\gamma'}{k\rho}\dot{w}_r \tag{6d}$$

$$K\frac{V_c^2}{r}\frac{\partial}{\partial \theta}(\alpha e - e') = N\ddot{u}_\theta + \frac{N}{f}\ddot{w}_\theta + \frac{\gamma'}{k\rho}\dot{w}_\theta \tag{6e}$$

$$KV_c^2\frac{\partial}{\partial z}(\alpha e - e') = N\ddot{u}_z + \frac{N}{f}\ddot{w}_z + \frac{\gamma'}{k\rho}\dot{w}_z, \tag{6f}$$

where $N$ designates the dimensionless ratio $N = \rho'/\rho$, $K = \alpha'/(2\mu+\lambda)$ and $V_c$, $V_s$ stand for the velocities of the compressional and shear waves, respectively, given by

$$V_c^2 = (2\mu+\lambda)/\rho, \quad V_s^2 = \mu/\rho. \tag{7}$$

In eqns (6a–f), the rotations of an element of the solid skeleton have been denoted by $\Omega_r$, $\Omega_\theta$, $\Omega_z$, where

$$2\Omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \tag{8a}$$

$$2\Omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{8b}$$

$$2\Omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right]. \tag{8c}$$

## 3. DILATATIONAL AND SHEAR WAVES

In order to determine the equations governing the propagation of the dilatational and shear waves, it is convenient to use the following displacement representations (Wolf, 1985)

$$u_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial \Omega}{\partial \theta} + \frac{\partial^2 \Psi}{\partial r\,\partial z} \tag{9a}$$

$$u_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \Psi}{\partial \theta\,\partial z} - \frac{\partial \Omega}{\partial r} \tag{9b}$$

$$u_z = \frac{\partial \Phi}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} \tag{9c}$$

for the solid particles, and

$$w_r = \frac{1}{r}\frac{\partial \Phi'}{\partial r} + \frac{1}{r}\frac{\partial \Omega'}{\partial \theta} + \frac{\partial^2 \Psi'}{\partial r\,\partial z} \tag{10a}$$

$$w_\theta = \frac{1}{r}\frac{\partial \Phi'}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \Psi'}{\partial \theta\,\partial z} - \frac{\partial \Omega'}{\partial r} \tag{10b}$$

$$w_z = \frac{\partial \Phi'}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi'}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \Psi'}{\partial \theta^2} \tag{10c}$$

for the fluid particles. It follows that

$$e = \nabla^2\Phi \tag{11a}$$

$$e' = -\nabla^2\Phi' \tag{11b}$$

$$2\left(\frac{\partial \Omega_\theta}{\partial z} - \frac{1}{r}\frac{\partial \Omega_z}{\partial \theta}\right) = \frac{\partial^2}{\partial r\,\partial z}(\nabla^2\Psi) + \frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2\Omega) \tag{11c}$$

$$2\left(\frac{\partial \Omega_z}{\partial r} - \frac{\partial \Omega_r}{\partial z}\right) = \frac{1}{r}\frac{\partial^2}{\partial \theta\,\partial z}(\nabla^2\Psi) - \frac{\partial}{\partial r}(\nabla^2\Omega) \tag{11d}$$

$$2\left(\frac{1}{r}\frac{\partial \Omega_r}{\partial \theta} - \frac{\partial \Omega_\theta}{\partial r} - \frac{\Omega_\theta}{r}\right) = -\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\nabla^2\Psi) - \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(\nabla^2\Psi)\right], \tag{11e}$$

in which $\nabla^2$ designates the Laplacian operator in cylindrical coordinates, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{12}$$

Upon inserting eqns (10) and (11) in (7), the dilatational waves (P-waves) are found to be governed by the solutions of the equations

$$V_c^2\nabla^2(\Phi+\alpha K\Phi') = \ddot{\Phi}+N\ddot{\Phi}' \tag{13a}$$

$$KV_c^2\nabla^2(\alpha\Phi+\Phi') = N\ddot{\Phi}+\frac{N}{f}\ddot{\Phi}'+\frac{\gamma'}{k\rho}\dot{\Phi}' \tag{13b}$$

while the horizontal shear waves (SH-waves) are determined from

$$V_s^2\nabla^2\Omega = \ddot{\Omega}+N\ddot{\Omega}' \tag{14a}$$

$$N\ddot{\Omega}+\frac{N}{f}\ddot{\Omega}'+\frac{\gamma'}{k\rho}\dot{\Omega}' = 0 \tag{14b}$$

and the vertical shear waves (SV-waves) from

$$V_s^2\nabla^2\Psi = \ddot{\Psi}+N\ddot{\Psi}' \tag{15a}$$

$$N\ddot{\Psi}+\frac{N}{f}\ddot{\Psi}'+\frac{\gamma'}{k\rho}\dot{\Psi}' = 0. \tag{15b}$$

## 4. ROCKING VIBRATIONS

For rocking vibrations (rotations about the $y$-axis, $y = r\sin\theta$) with constant amplitude, $\delta_R$, induced by harmonically varying moment, $M\mathrm{e}^{\mathrm{i}\omega t}$, applied to the circular disk, the boundary conditions on $z = 0$ are:

$$p(r,\theta,0:t) = \tau_{rz}(r,\theta,0:t) = \tau_{\theta z}(r,\theta,0:t) = 0, \quad r \geqslant 0, \quad \text{all } \theta \tag{16a}$$

$$u_z(r,\theta,0:t) = \delta_R r\cos\theta\,\mathrm{e}^{\mathrm{i}\omega t}, \quad 0 \leqslant r \leqslant a, \quad \text{all } \theta \tag{16b}$$

$$\tau_z(r,\theta,0;t) = 0, \quad r \geqslant 0, \quad \text{all } \theta. \tag{16c}$$

An appropriate integral transform solution of eqns (13)–(15) is obtained by writing

$$\Phi = \int_0^\infty [A_1(s)\,\mathrm{e}^{-n_1 z}+A_2(s)\,\mathrm{e}^{-n_2 z}]J_1(rs)\,\mathrm{d}s\cos\theta\,\mathrm{e}^{\mathrm{i}\omega t} \tag{17a}$$

$$\Phi' = \int_0^\infty [\beta_1 A_1(s)\,\mathrm{e}^{-n_1 z}+\beta_2 A_2(s)\,\mathrm{e}^{-n_2 z}]J_1(rs)\,\mathrm{d}s\cos\theta\,\mathrm{e}^{\mathrm{i}\omega t} \tag{17b}$$

$$\begin{Bmatrix}\Omega\\ \Psi\end{Bmatrix} = \int_0^\infty \begin{Bmatrix}A_3(s)\\ A_4(s)\end{Bmatrix}[\mathrm{e}^{-n_3 z}J_1(rs)\,\mathrm{d}s]\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}\mathrm{e}^{\mathrm{i}\omega t} \tag{17c}$$

$$\begin{Bmatrix}\Omega'\\ \Psi'\end{Bmatrix} = \beta_3\begin{Bmatrix}\Omega\\ \Psi\end{Bmatrix} \tag{17d}$$

in which

$$n_j = (s^2-\omega H_j^2)^{1/2}, \quad \mathrm{Re}\,n_j \geqslant 0, \quad \mathrm{Im}\,n_j \geqslant 0, \tag{18}$$

and $H_j$, $\beta_j$, $j = 1, 2, 3$, are known functions of the material properties and the applied frequency and given in the Appendix. Moreover, in eqns (17a–d), $J_1$ denotes the Bessel function of the first kind of order unity, and $A_j(s)$, $j = 1, 2, 3$ and 4 are transform parameters to be determined from the applicable boundary conditions. For convenience, the factor $\mathrm{e}^{\mathrm{i}\omega t}$ containing the frequency of excitations will be omitted from all applicable equations in the remaining parts of the paper. The expressions of the displacements and stresses throughout the medium are readily obtained by inserting eqns (17) and (18) into (4), (5), (9) and (10).

Boundary conditions (16a) are readily shown to imply that

$$A_2(s) = \gamma_1 A_1(s)$$
$$A_3(s) = 0$$
$$A_4(s) = \gamma_2 A_1(s) \tag{19}$$

in which $\gamma_1$ and $\gamma_2$ stand for the abbreviations

$$\gamma_1 = -(\alpha+\beta_1)H_1^2/(\alpha+\beta_2)H_2^2 \tag{20a}$$

$$\gamma_2 = 2(n_1+\gamma_1 n_2)/(n_3^2+s^2). \tag{20b}$$

With a view towards establishing the dual integral equations governing the remaining unknown function, $A_1(s)$, the following abbreviation is introduced

$$A(s) = D_1(\omega, v)[(1+\gamma_1)s^2 - s^2 n_3\gamma_2 + V_c^2(1-\alpha^2 K)\omega H_1^2(\beta_1-\beta_2)/(2V_s^2)(\alpha+\beta_2)]A_1(s), \tag{21}$$

where $D_1(\omega, v)$ is an arbitrary function introduced for convenience and determined in the following. Observing that

$$(\lambda+2\mu)/(2\mu) = V_c^2/(2V_s^2)$$
$$\alpha'/(2\mu) = KV_c^2/(2V_s^2) \tag{22}$$

it follows from applying boundary conditions (16b) and (16c) to the appropriate expressions for the stresses and displacements that

$$\int_0^\infty s^{-1}[1+F(s,\omega)]A(s)J_1(rs)\,\mathrm{d}s = \delta_R r, \quad r < a, \tag{23a}$$

$$\int_0^\infty A(s)J_1(rs)\,\mathrm{d}s = 0, \qquad r > a, \tag{23b}$$

where the kernel, $F(s, \omega)$, is given by

$$F(s,\omega) = \frac{s(\gamma_2 s^2 - n_1 - \gamma_1 n_2)}{D_1\{(1+\gamma_1)s^2 - s^2 n_3\gamma_2 + V_c^2(1-\alpha^2 K)\omega H_1^2(\beta_1-\beta_2)/[(2V_s^2)(\alpha+\beta_2)]\}} - 1. \tag{24}$$

Since in the static case the function $F(s, \omega)$ must vanish, it follows upon expanding $F(s, \omega)$ for small values of $\omega$ that

$$D_1(s, v) = \frac{H_3^2[(\alpha+\beta_2)H_2^2-(\alpha+\beta_1)H_1^2]}{H_1^2 H_2^2(\beta_1-\beta_2)[V_c^2(1-\alpha^2 K)/V_s^2-1]}. \tag{25}$$

Note that in the one-phase material, $D_1$ reduces to $D_1 = -2(1-v)$. Also, it is readily shown that the function $F(s, \omega)$ is bounded for large values of the parameters. The solution of the set of equations (23a, b) is readily obtained by writing

$$A(s) = \frac{(2s)}{\pi}\int_0^a Q(t)\sin(st)\,\mathrm{d}t, \tag{26}$$

where the auxiliary function, $Q(t)$, satisfies the integral equation

$$Q_1(t)+\frac{2}{\pi}\int_0^a L(t,u)Q(u)\,\mathrm{d}u = 2t\delta_R \tag{27}$$

with

$$L(t,u) = \int_0^\infty F(s,\omega)\sin(su)\sin(st)\,\mathrm{d}s. \tag{28}$$

The total moment under the indentor is given by

$$M = -\int_0^{2\pi}\cos(\theta)\,\mathrm{d}\theta\int_0^a r^2\tau_z\,\mathrm{d}r$$

which may be shown upon using eqns (4c), (9), (10), (17), (21) and (26) to reduce to

$$M = -(8\mu)/D_1\int_0^a yQ(y)\,\mathrm{d}y. \tag{29}$$

Since in the one-phase static case, $D_1 = -2(1-\nu)$ and $Q(y) = 2(\delta_{R_{\text{stat}}})y$, it follows that the ratio of the overturning moments is

$$M/M_{\text{stat}} = -[3(1-\nu)/(a^3D_1)]\int_0^a yQ(y)\,\mathrm{d}y(\delta_R/\delta_{R_{\text{stat}}}). \tag{30}$$

The impedance functions, $m_1$ and $m_2$, given by the following equation

$$(M/M_{\text{stat}})\,\mathrm{e}^{-i\omega t} = (m_1+\mathrm{i}m_2)(\delta_R/\delta_{R_{\text{stat}}}), \tag{31}$$

are computed for a wide range of values of the dimensionless frequency parameter, $\varpi = a\omega/V_s$. A medium modeling water-saturated sand with the following properties is assumed to occupy the half-space: $\alpha = 1.0$, $\alpha' = 2.939\times10^5$ psi (2026 MPa), $\rho = 0.01937\times10^{-2}$ lb s$^2$ in$^{-4}$ (2070 kg m$^{-3}$), $\rho' = 0.0327\times10^{-3}$ lb s$^2$ in$^{-4}$ (350 kg m$^{-3}$), $f = 0.35$ and the confined modulus $E_c = (1-\nu)\lambda/\nu = 14\times10^3$ psi (96.5 MPa). In order to explore the influence of permeability on the impedance functions, three values of the non-dimensional permeability coefficient, $k' = (V_s/ag)k$, namely, $k' = 0.001$, 0.01 and 1.0, are used in the numerical evaluations. Here, $g$ is the gravitational acceleration.

Figure 1 shows the variation of the impedance functions, $m_1$ and $m_2$, with $\tilde{\omega}$ for $k' = 1.0$ and $\nu = 1/3$. The top curve represents the real part ($m_1$) of the moment–angular displacement relationship (stiffness) while the lower curve represents the imaginary part normalized by $\tilde{\omega}$ (damping coefficient). In order to compare the results with the dry material, the corresponding results from the work of Luco and Westmann (1971) are shown as well. As might be expected, the response of the two-phase medium with $k' = 1.0$ (relatively high permeability coefficient) is similar to that of the dry solid. The stiffness of the saturated medium is reduced for $\tilde{\omega} > 5$, while there is no change in the damping coefficient over the range of the frequency parameter.

Figure 2 reveals the influence of permeability on the rocking impedance. For $\tilde{\omega} > 3$, the stiffness is significantly more sensitive to the values of the permeability coefficient. The values of the damping coefficient are increased with lower values of $k'$. A typical saturated sand medium is represented by $k' = 0.01$ and for such material, the influence of the pore fluid on the impedance values is significant.

Figure 3 shows the influence of varying Poisson's ratios on the response of a medium with very small values of the permeability number ($k' = 0.001$).

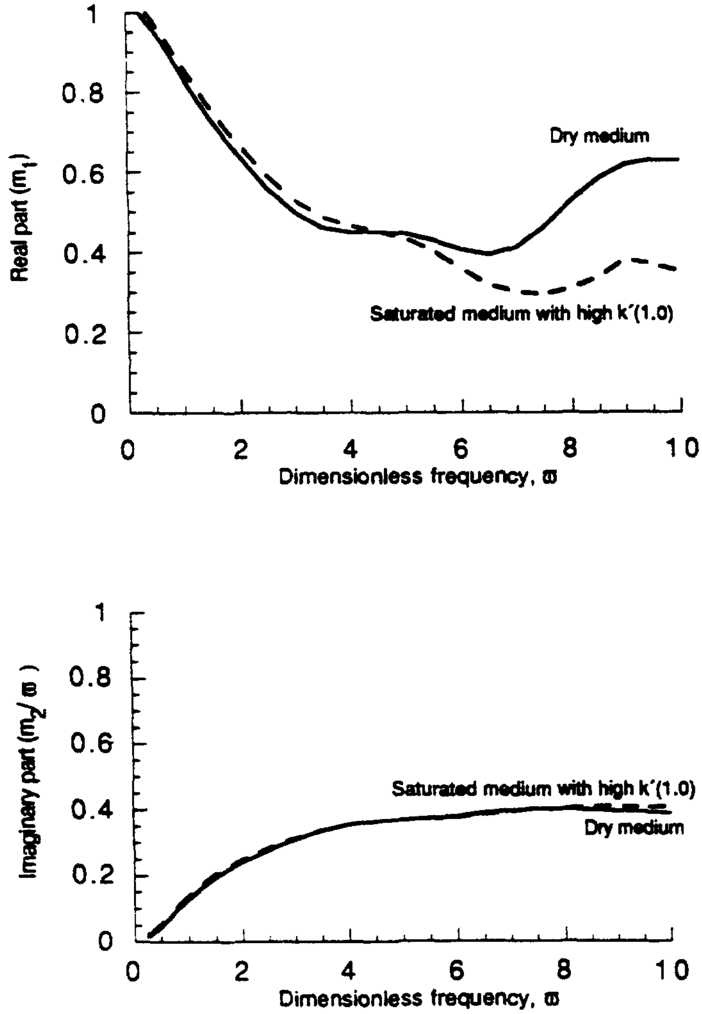


Fig. 1. Impedance variations with $\bar{\omega}$ for $v = 1/3$ and $k' = 1.0$.

## 5. HORIZONTAL VIBRATIONS

When the indentor experiences in-plane horizontal translation parallel to the $y$-axis, induced by a harmonically varying horizontal force, $H\mathrm{e}^{\mathrm{i}\omega t}$, the appropriate boundary conditions in the $z = 0$ plane are:

$$p(r, \theta, 0) = \tau_z(r, \theta, 0) = 0, \quad r > 0, \quad \text{all } \theta \tag{32a}$$

$$u_r(r, \theta, 0) = \delta_H \cos\theta, \quad 0 \leqslant r \leqslant a, \quad \text{all } \theta \tag{32b}$$

$$u_\theta(r, \theta, 0) = -\delta_H \sin\theta, \quad 0 \leqslant r \leqslant a, \quad \text{all } \theta \tag{32c}$$

$$\tau_{rz}(r, \theta, 0) = \tau_{\theta z}(r, \theta, 0) = 0, \quad r > 0, \quad \text{all } \theta. \tag{32d}$$

In this case, the expressions in eqns (17a–d) are also used for the dilatational and shear waves. However, the boundary conditions (32a) in conjunction with the appropriate expressions for the pore pressure $p$ and normal stress $\tau_z$ developed from eqns (9), (10) and (17) yield

$$\begin{aligned} A_2(s) &= \gamma_1 A_1(s) \\ A_3(s) &\neq 0 \\ A_4(s) &= \gamma_3 A_1(s) \end{aligned} \tag{33}$$

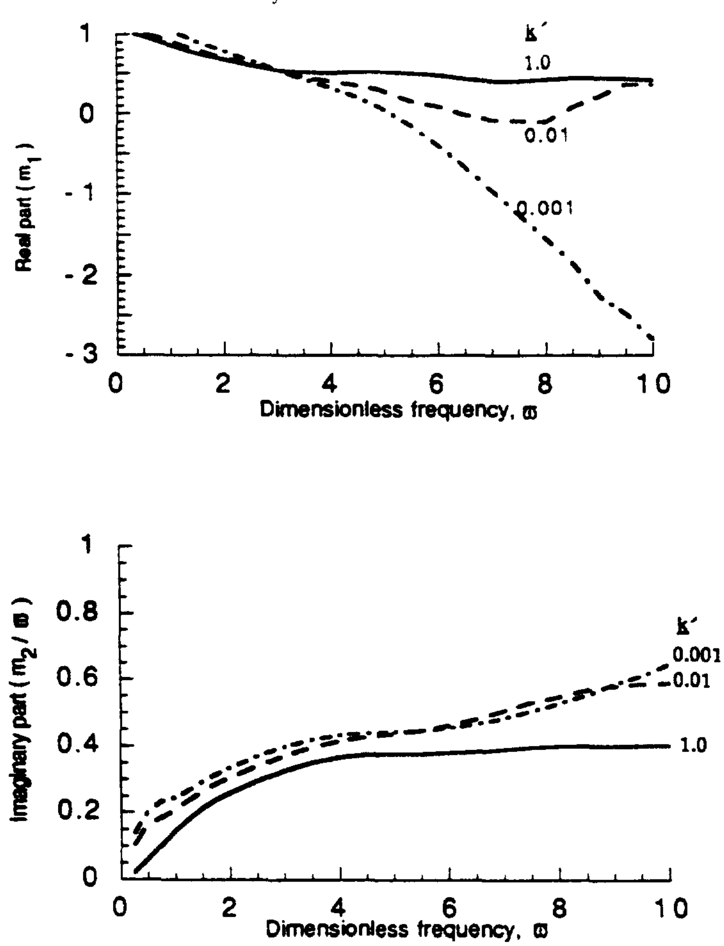


Fig. 2. Influence of permeability on the rocking impedance ($\nu = 0.25$).

in which $\gamma_1$ is defined in eqn (20) and $\gamma_3$ is given by

$$2\mu n_3 s^2 \gamma_3 = 2\mu(n_1^2+\gamma_1 n_2^2) - \omega H_1^2(\lambda+\alpha\alpha'\beta_1) - \omega H_2^2\gamma_1(\lambda+\alpha\alpha'\beta_2), \tag{34}$$

where the same notations for $n_j$ and $H_j$, $j = 1, 2, 3$, are used (see the Appendix). Introducing the abbreviations

$$\begin{aligned} B_1(s) &= D_2(\omega,\nu)[2n_1+2\gamma_1 n_2-(n_3^2+s^2)\gamma_3]sA_1(s) \\ B_2(s) &= D_2(\omega,\nu)[n_3 s]A_3(s), \end{aligned} \tag{35}$$

where $D_2$ is an arbitrary function, it follows that eqns (9), (10), (17) and (33) yield the following expressions for the displacements $u_r$ and $u_\theta$ across the $z = 0$ plane

$$\begin{aligned} u_r(r,\theta,0) = \frac{1}{2}\int_0^\infty & \left\{\left\langle [1+G_1(s)]B_1(s)+[1+G_2(s)]\frac{B_2(s)}{D_2}\right\rangle \frac{J_0(rs)}{s}\right. \\ & \left. +\left\langle -[1+G_1(s)]B_1(s)+[1+G_2(s)]\frac{B_2(s)}{D_2}\right\rangle \frac{J_2(rs)}{s}\right\} \mathrm{d}s \cos\theta \end{aligned} \tag{36a}$$

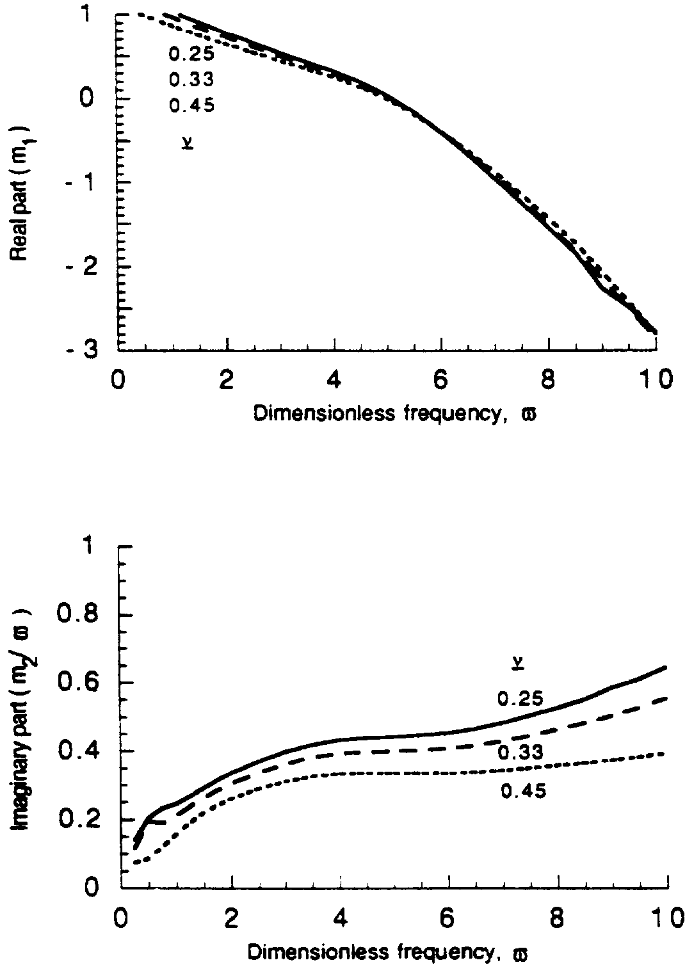


Fig. 3. Influence of Poisson's ratio on the rocking impedance ($k' = 0.001$).

$$u_\theta(r,\theta,0) = \frac{1}{2}\int_0^\infty \left\{\left\langle -[1+G_1(s)]B_1(s) - [1+G_2(s)]\frac{B_2(s)}{D_2}\right\rangle \frac{J_0(rs)}{s} + \left\langle -[1+G_1(s)]B_1(s) + [1+G_2(s)]\frac{B_2(s)}{D_2}\right\rangle \frac{J_2(rs)}{s}\right\} \mathrm{d}s \sin\theta. \tag{36b}$$

Similarly, the expressions for the shearing stresses are obtained by using eqns (4), (9), (17) and (33). The results for $z = 0$ are

$$\tau_{rz}(r,\theta,0) = \frac{\mu}{2D_2}\int_0^\infty \{[-B_1(s) - B_2(s)]J_0(rs) + [B_1(s) - B_2(s)]J_2(rs)\}\,\mathrm{d}s\cos\theta \tag{37a}$$

$$\tau_{\theta z}(r,\theta,0) = \frac{\mu}{2D_2}\int_0^\infty \{[B_1(s) + B_2(s)]J_0(rs) + [B_1(s) - B_2(s)]J_2(rs)\}\,\mathrm{d}s\sin\theta. \tag{37b}$$

In eqns (36) and (37), $J_n$, $n = 0,2$, are the Bessel functions of the first kind and order $n$, and $G_1$ and $G_2$ stand for

$$G_1(s) = \frac{s(1+\gamma_1 - n_3\gamma_3)}{D_2[2n_1 + 2n_2\gamma_1 - (n_3^2 + s^2)\gamma_3]} - 1 \tag{38a}$$

$$G_2(s) = \frac{s}{n_3} - 1, \tag{38b}$$

where $D_2(\omega,\nu)$ is chosen to ensure that $G_1 \to 0$ as $\omega \to 0$. In this manner it follows that

$$D_2(\omega, \nu) = K(\alpha K - 1)\left(\frac{V_c^2}{2V_s^2}\right)\left[\alpha^2 K^2 + (1-\alpha K)\left(1 - \frac{V_c^2}{2V_s^2}\right)\right]^{-1}. \tag{39}$$

For the one-phase elastic material, $D_2$ reduces to $D_2 = 1 - \nu$. Also, $G_1$ and $G_2$ are bounded for large values of $s$.

Equations (36) and (37), in conjunction with boundary conditions (32b–d), yield the following equations for the determination of the unknown functions $B_1(s)$ and $B_2(s)$:
For $0 \leqslant r \leqslant a$ and all $\theta$:

$$\int_0^\infty [D_2(1+G_1)B_1(s) + (1+G_2)B_2(s)]\frac{J_0(rs)}{s}\,\mathrm{d}s = 2D\delta_H \tag{40a}$$

$$\int_0^\infty [D_2(1+G_1)B_1(s) - (1+G_2)B_2(s)]\frac{J_2(rs)}{s}\,\mathrm{d}s = 0 \tag{40b}$$

and for $r > a$, all $\theta$:

$$\int_0^\infty [B_1(s) + B_2(s)]J_0(rs)\,\mathrm{d}s = 0 \tag{41a}$$

$$\int_0^\infty [B_1(s) - B_2(s)]J_2(rs)\,\mathrm{d}s = 0. \tag{41b}$$

Dual integral equations similar to eqns (40) and (41) were considered by Gladwell (1969) and reduced to coupled Fredholm integral equations of the second kind. The real and imaginary parts of the impedance were obtained by expanding the kernels and the unknown functions of the integral equations in power series valid for small values of the wave number. The impedances for large values of the wave number were not obtained. The same equations were also considered by Luco and Westmann (1971), and Bielak (1971) and reduced to coupled integral equations amenable to numerical treatment. Numerical values of the impedance components were then obtained for a wide range of values of the applied frequency. A procedure similar to that used by Bielak (1971) is applied here.

With a view towards computing numerical values of the impedance functions, let

$$B_1(s) = s\int_0^a R_1(t)\cos(st)\,\mathrm{d}t \tag{42a}$$

$$B_2(s) = \left(\frac{1}{a}\right)\left[\sin(as) + \int_0^a (st)^{3/2}R_2(t)J_{3/2}(st)\,\mathrm{d}t\right]\int_0^a R_1(t)\,\mathrm{d}t \tag{42b}$$

provided that the limiting values of $R_1(t)$ and $R_2(t)$ as $t \to 0$ are finite. In eqns (42a, b), $J_{3/2}$ is the Bessel function of the first kind of order 3/2 and $R_1$ and $R_2$ are auxiliary functions. Upon performing an integration by parts on expressions (42), it may be shown that boundary conditions (41a,b) are automatically satisfied. In order to establish the Fredholm equations, rewrite eqn (40a) in the form

$$D_2\int_0^\infty B_2(s)\frac{J_0(rs)}{s}\,\mathrm{d}s = 2D_2\delta_H - D_2\int_0^\infty G_1(s)B_1(s)\frac{J_0(rs)}{s}\,\mathrm{d}s$$
$$-\int_0^\infty [1+G_1(s)]B_2(s)\frac{J_0(rs)}{s}\,\mathrm{d}s. \tag{43}$$

Inserting expressions (42a, b) into (43) leads to an integral equation of Abel's type whose solution yields

$$R_1(t)+\int_0^a R_1(y)K_{11}(y,t)\,\mathrm{d}y = \frac{4}{\pi}\delta_H-\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{aD_2}\left[\left(\frac{\pi}{2}\right)^{1/2}\right.$$

$$+\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty G_2(s)\frac{\sin(as)}{s}\cos(st)\,\mathrm{d}s-tR_2(t)$$

$$\left.+\int_t^a R_2(y)\,\mathrm{d}y+\int_0^a yR_2(y)K_{12}(y,t)\,\mathrm{d}y\right]\int_0^a R_1(y)\,\mathrm{d}y, \quad (44a)$$

where

$$K_{11}(y,t) = \frac{2}{\pi}\int_0^\infty G_1(s)\sin(st)\cos(sy)\,\mathrm{d}s \quad (44b)$$

$$K_{12}(y,t) = \frac{2}{\pi}\int_0^\infty G_2(s)\cos(st)\left[\frac{\sin(sy)}{sy}-\cos(sy)\right]\mathrm{d}s. \quad (44c)$$

Similarly, when eqns (42a,b) are used in conjunction with eqn (40b), the following Fredholm integral equation is obtained:

$$\frac{1}{a}\left\{\left(\frac{\pi}{2}\right)^{1/2}+R_2(t)+\int_0^a yR_2(y)K_{22}(y,t)\,\mathrm{d}y\right.$$

$$\left.+\int_0^\infty G_2(s)\frac{\sin(as)}{s}\left[\frac{\sin(st)}{st}-\cos(st)\right]\mathrm{d}s\right\}\int_0^a R_1(y)\,\mathrm{d}y$$

$$=\frac{\pi}{2}D_2\left[\frac{1}{t}\int_0^t R_1(y)\,\mathrm{d}y-R_1(t)+\int_0^a R_1(y)K_{21}(y,t)\,\mathrm{d}y\right] \quad (45a)$$

with

$$K_{22}(y,t) = \frac{2}{\pi}\int_0^\infty G_2(s)\left[\frac{\sin(sy)}{sy}-\cos(sy)\right]\left[\frac{\sin(st)}{st}-\cos(st)\right]\mathrm{d}s \quad (45b)$$

$$K_{21}(y,t) = \frac{2}{\pi}\int_0^\infty G_1(s)\left[\frac{\sin(st)}{st}-\cos(st)\right]\cos(sy)\,\mathrm{d}s. \quad (45c)$$

A simplified form of eqn (45) can be obtained as follows. First, eqn (44a) is integrated with respect to the variable $t$, between limits $(0, t)$, and the resulting equation and eqn (44a) are substituted in eqn (45a) to yield

$$R_2(t)+\int_0^a R_2(y)K_{22}(y,t)\,\mathrm{d}y = -\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty G_2(s)\left[\frac{\sin(st)}{st}-\cos(st)\right]\frac{\sin(as)}{s}\mathrm{d}s. \quad (46)$$

Equations (44) and (46) determine the auxiliary functions $R_1(t)$ and $R_2(t)$ either numerically or by approximation.

The horizontal force applied to the indentor, $He^{i\omega t}$, in the $\theta = 0$ direction has an amplitude given by

$$H = \int_0^{2\pi} \int_0^a [-\tau_{rz}(r,\theta,0)\cos\theta + \tau_{\theta z}(r,\theta,0)\sin\theta] r \, dr \, d\theta \tag{47}$$

which may be shown to simplify to

$$H = \frac{2\pi\mu}{D_2} \int_0^a R_1(y)\, dy. \tag{48}$$

In the static ($\omega \to 0$) one-phase material, $G_2(s)$ in eqn (38) vanishes and it is found that $R_1(t) = (4/\pi)[(1-\nu)/(2-\nu)]\delta_{H_{stat}}$, $R_2(t) = 0$. It follows that the expression in eqn (48) reduces to

$$H_{stat} = \left(\frac{8\mu a}{2-\nu}\right)\delta_{H_{stat}}. \tag{49}$$

This is as far as the solution can be analytically developed. With a view towards determining the influence of pore fluid on the response of the half-space, eqns (44) and (46) are discretized by transforming them to a system of simultaneous, linear, algebraic equations to yield the values of $R_1(t)$ and $R_2(t)$, and numerical values of the ratio

$$\left(\frac{H}{H_{stat}}\right) e^{-i\omega t} = (h_1 + ih_2)\left(\frac{\delta_H}{\delta_{H_{stat}}}\right) \tag{50}$$

are determined. The same material properties as those used in the rocking case are considered.

The variations of the functions $h_1$ and $h_2/\bar{\omega}$ with $\bar{\omega}$ are shown in Figs 4–6. Figure 4 compares the horizontal impedance data for a two-phase medium ($k' = 1.0$) with the dry medium obtained by Luco Westmann (1971). In the range $\bar{\omega} < 2.0$, the magnitudes of these functions are identical in both media. However, for $\bar{\omega} > 2.0$ there is about 20–25% variation in the impedance values due to the presence of the pore fluid. Figure 5 shows the variations of $h_1$ and $h_2/\bar{\omega}$ with $k'$. It is clear that the horizontal impedance functions are insensitive to variations in the permeability coefficient. Similar conclusions were reached by Halpren and Christiano (1986) for a square plate supported by a porous elastic solid. Figure 6 reveals the influence of varying Poisson's ratio on the impedance functions ($k' = 0.001$). Increasing the values of Poisson's ratio lowers the magnitude of the stiffness and damping coefficient of the medium.

## 6. DISCUSSION AND CONCLUSIONS

Mixed-boundary value problems were formulated to determine the response of a pervious circular footing supported at the surface of a two-phase elastic medium and excited by a harmonically varying overturning moment and a horizontal force. By using the integral transform technique and auxiliary functions, the problems were reduced to Fredholm integral equations of the second kind which, in turn, were solved numerically to generate the impedance functions (stiffness and damping coefficients). Numerical data of the impedance functions have been computed for a wide range of values of the applied frequency to reveal the influences of pore fluid and Poisson's ratio of the solid skeleton. For the rocking vibration case, the presence of the pore fluid significantly affects the magnitudes of the impedance parameters (both magnitude and sign) while in the horizontal vibration case, the presence of the pore fluid marginally influences the impedance response. It should be emphasized that the solutions presented in this paper are applicable only to the completely pervious surface condition. Other drainage conditions such as partially drained and completely undrained surface conditions are also important from the application viewpoint and will be addressed in future studies.

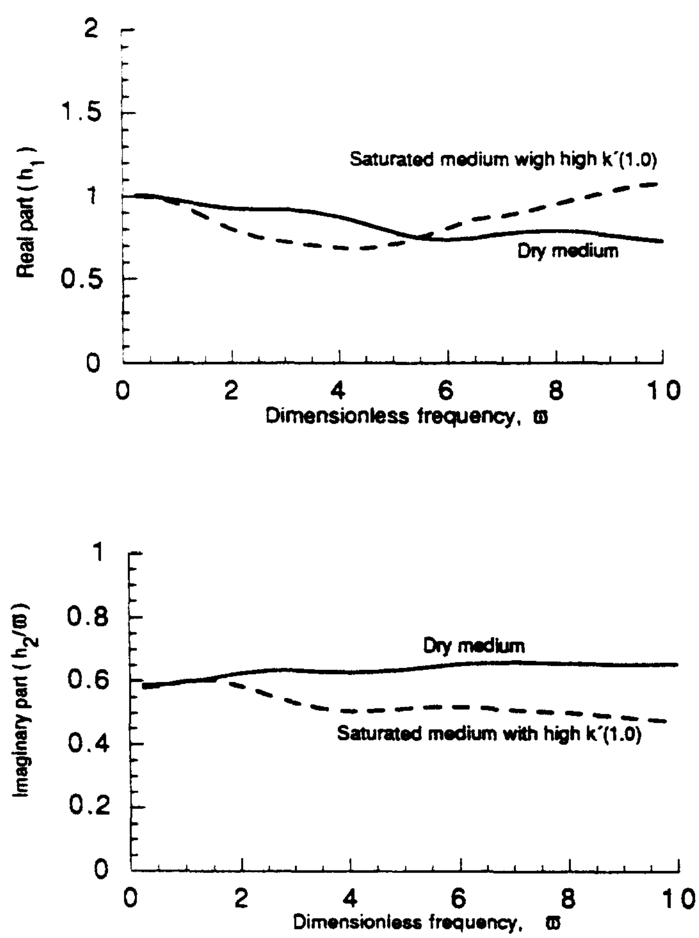


Fig. 4. Impedance variations with $\bar{\omega}$ for $\nu = 1/3$ and $k' = 1.0$.

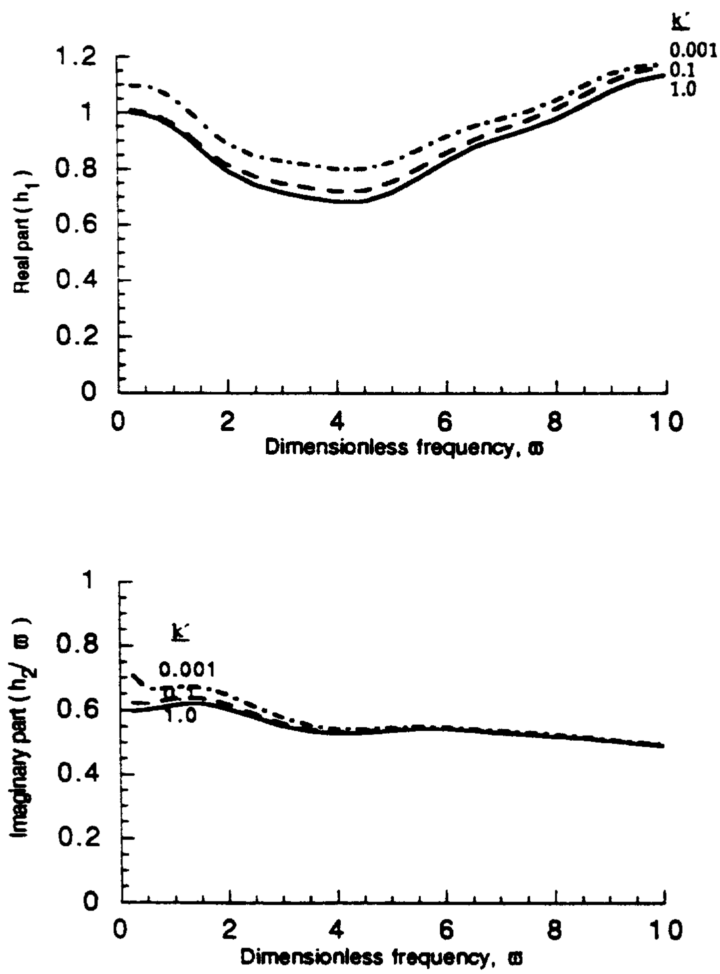


Fig. 5. Influence of permeability on the horizontal impedance ($\nu = 0.25$).

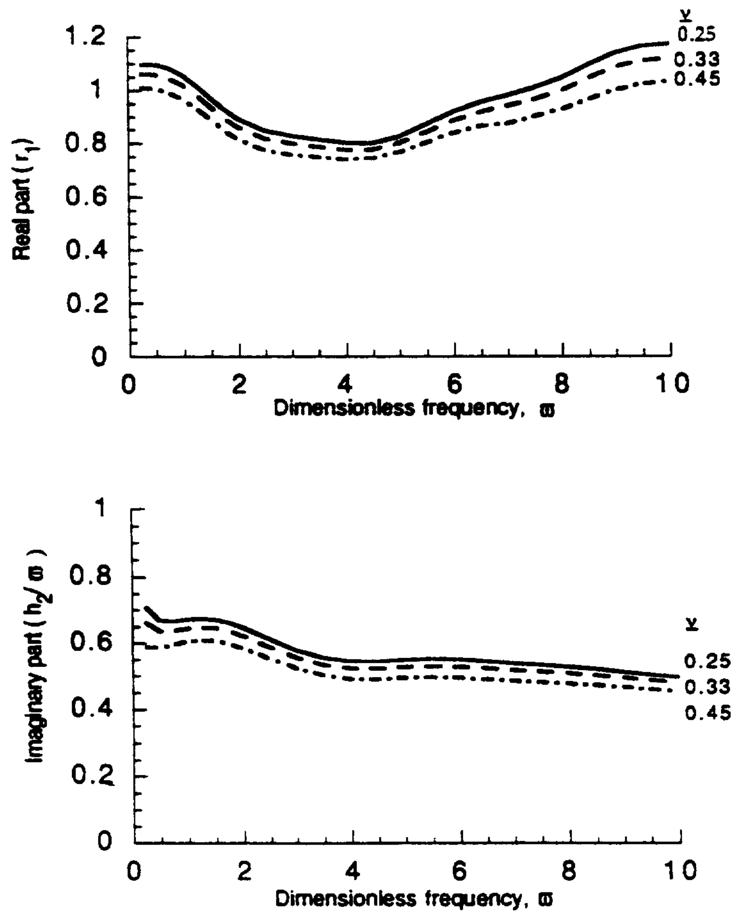


Fig. 6. Influence of Poisson's ratio on the horizontal impedance ($k' = 0.001$).

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## APPENDIX: ABBREVIATIONS

The following abbreviations are used in the paper:

$$H_{1,2}^2 = \frac{1}{2V_c d}[-b \pm (b^2 - 4dc)^{1/2}] \tag{A1}$$

$$d = K(\alpha^2 K - 1) \tag{A2}$$

$$b = \frac{i\gamma'}{k\rho} + \left(2\alpha KN - K - \frac{N}{f}\right)\omega \tag{A3}$$

$$c = \left(\frac{i\gamma'}{k\rho}\right)\omega + \left(N^2 - \frac{N}{f}\right)\omega^2 \tag{A4}$$

$$H_3^2 = \frac{c}{V_s^2[(i\gamma'/k\rho) - (N/f)\omega]} \tag{A5}$$

$$\beta_j = \frac{(\alpha N - 1) + (1 - \alpha^2 K)V_c^2 H_j^2}{(i\alpha\gamma'/k\rho) + [N - (\alpha N/f)]\omega}, \quad j = 1, 2 \tag{A6}$$

$$\beta_3 = \frac{N\omega}{(i\gamma'/k\rho) - (N/f)\omega}. \tag{A7}$$